

SYMPLECTIC EMBEDDINGS, THE PREPOTENTIALS AND AUTOMORPHIC FUNCTIONS OF $\frac{SU(1, n)}{U(1) \otimes SU(n)}$

W. A. SABRA^{a, *}, S. THOMAS^{b, †} AND N. VANEGAS^{b, ‡}

^a *Physics Department, Royal Holloway and Bedford New College,
University of London, Egham Surrey,
TW20 OEX, United Kingdom.*

^b *Physics Department,
Queen Mary and Westfield College,
Mile End Road, London E1 4NS,
United Kingdom.*

ABSTRACT

The holomorphic homogeneous prepotential encoding the special geometry of the special Kähler manifolds $\frac{SU(1, n)}{U(1) \otimes SU(n)}$ is constructed using the symplectic embedding of the isometry group $SU(1, n)$ into $Sp(2n + 2, \mathbf{R})$. Also the automorphic functions of these manifolds are discussed.

★ e-mail: uhap012@vax.rhbnc.ac.uk

† e-mail: s.thomas@qmw.ac.uk

‡ e-mail: n.vanegas@qmw.ac.uk

Special geometry has emerged as an important structure in the study of supergravity and superstring theories [1-13]. This geometric structure arises in the study of the moduli spaces of $N = 1$ compactifications on Calabi-Yau manifolds, $N = 2$ supergravity theory and appears as a consequence of $(2, 2)$ worldsheet superconformal field theory or $N = 2$ supersymmetry in the target space.

The concept of special Kähler geometry has first appeared in the physics literature in the study of the coupling of an n vector multiplets to four dimensional $N = 2$ supergravity [2,3,4]. The lagrangian for the coupling of the vector multiplets was derived using the

superconformal tensor calculus [2]. In addition to the hypermultiplets, the theory is described by introducing $(n + 1)$ vector multiplets with scalar components X^I , $(I = 0, \dots, n)$, where the extra multiplet labelled by 0 contains the graviphoton. The couplings were found to be described in terms of a holomorphic function F , referred to as the prepotential, of degree two in terms of the scalar fields defining an n -dimensional complex hypersurface defined by the gauge fixing condition

$$i(X^I \bar{F}_I - F_I \bar{X}^I) = 1. \quad (1)$$

Letting the X^I be proportional to holomorphic sections $Z^I(z)$ of a projective $(n + 1)$ -dimensional space, where z is a set of n complex coordinates, then the z coordinates parametrize a Kähler space with metric $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K$, where K , the Kähler potential is expressed by

$$\begin{aligned} K &= -\log i(Z^I \bar{F}_I(\bar{Z}) - F_I(Z) \bar{Z}^I) \\ &= -\log \left(i \langle \Omega | \bar{\Omega} \rangle \right) = -\log i \begin{pmatrix} Z^I & F_I(Z) \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \bar{Z}^I \\ \bar{F}_I(\bar{Z}) \end{pmatrix}, \quad (2) \\ X^I &= e^{K/2} Z^I. \end{aligned}$$

Special coordinates correspond to the choice

$$z^\alpha = \frac{X^\alpha}{X^0}; \quad Z^0(z) = 1, \quad Z^\alpha(z) = z^\alpha. \quad (3)$$

The gauge kinetic terms, expressed in terms of the matrix \mathcal{N} , corresponding

to the $(n + 1)$ vector are given by

$$\mathcal{N}_{ij} = \bar{F}_{ij} + 2i \frac{(Im F_{im})(Im F_{jn})X^m X^n}{(Im F_{ab})X^a X^b}. \quad (4)$$

Homogeneous symmetric special manifolds were classified in [5] with their corresponding holomorphic prepotentials. In addition to four sporadic cases, the list of special Kähler manifolds includes two series, one of which is given in the form of a direct product, which have been proved to be the only allowed case [6]. These two series are given by

$$\begin{aligned} SK(n+1) &= \frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2) \otimes SO(n)}, \\ CP_{n-1,1} &= \frac{SU(1,n)}{SU(n) \otimes U(1)}. \end{aligned} \quad (5)$$

The classical holomorphic prepotentials for the manifolds $SK(n+1)$ have been analyzed in [14]. There, the classical holomorphic function F was obtained in the spirit of the method of [15] in constructing the lagrangian of an abelian gauge theory with n gauge fields and a set of scalar fields parametrizing a coset space. In this construction, the duality transformations on the gauge sector of the theory are parametrized by the embedding of the isometry group of the coset space of the scalar fields into the group $Sp(2n+2, \mathbf{R})$ or $Usp(n, n)$. Such an embedding can then be used to determine the dependence of the gauge kinetic terms on the scalar fields. In [14], the section (X^Λ, F_Λ) is introduced as a symplectic vector. A relation between F_Λ and X^Λ is then determined in an appropriate embedding of the isometries of $SK(n+1)$ into $Sp(2n+2, \mathbf{R})$.

In this letter, the generalization of the results of [14] to the second infinite series of special manifolds $CP_{n-1,1}$, the so called minimal coupling, is performed. We construct their holomorphic prepotentials encoding their special geometry and discuss their automorphic functions which could serve as non-perturbative superpotentials for the scalar fields.

The isometry group of the cosets $CP_{n-1,1}$ is given by the group $SU(1, n)$. As a first step in the construction of the holomorphic function for these manifolds, we have to construct an embedding of their isometry group into the symplectic group $Sp(2n + 2, \mathbf{R})$.

We represent an element of $SU(1, n)$ by the complex $(n + 1) \times (n + 1)$ matrix \mathbf{M} satisfying the conditions

$$\mathbf{M}^\dagger \eta \mathbf{M} = \eta, \quad \det \mathbf{M} = 1, \quad (6)$$

where η is the constant diagonal metric with signature $(+, -, \dots, -)$. Decomposing the matrix \mathbf{M} into its real and imaginary parts,

$$\mathbf{M} = \mathbf{U} + i\mathbf{V}, \quad (7)$$

then from (6) we obtain, for the real matrices \mathbf{U} and \mathbf{V} , the following relations

$$\begin{aligned} \mathbf{U}^t \eta \mathbf{U} + \mathbf{V}^t \eta \mathbf{V} &= \eta, \\ \mathbf{U}^t \eta \mathbf{V} - \mathbf{V}^t \eta \mathbf{U} &= 0. \end{aligned} \quad (8)$$

Represent an element of $Sp(2n + 2, \mathbf{R})$ by the $(2n + 2) \times (2n + 2)$ real matrix $\mathbf{\Omega}$, this element satisfies

$$\mathbf{\Omega}^t \mathbf{L} \mathbf{\Omega} = \mathbf{L}, \quad \mathbf{L} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad (9)$$

If we write

$$\mathbf{\Omega} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \quad (10)$$

where the matrices \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} are $(n + 1) \times (n + 1)$ matrices, then in terms

of these component matrices, the condition in (9) implies the following conditions

$$\mathbf{A}^t \mathbf{C} - \mathbf{C}^t \mathbf{A} = \mathbf{0}, \quad \mathbf{A}^t \mathbf{D} - \mathbf{C}^t \mathbf{B} = \mathbf{I}, \quad \mathbf{B}^t \mathbf{D} - \mathbf{D}^t \mathbf{B} = \mathbf{0}. \quad (11)$$

An embedding of $SU(1, n)$ into the symplectic group $Sp(2n+2, \mathbf{R})$ is given by

$$\mathbf{A} = \mathbf{U}, \quad \mathbf{C} = -\eta \mathbf{V}, \quad \mathbf{B} = \mathbf{V} \eta, \quad \mathbf{D} = \eta \mathbf{U} \eta. \quad (12)$$

Having obtained an embedding $\mathbf{\Omega}_e$ with matrix components as given in (12), the next step is to introduce the symplectic section (X^Λ, F_Λ) , where $\Lambda = 0, \dots, n$, and demand that it transforms as a vector under the symplectic transformations induced by $\mathbf{\Omega}_e$. These transformation rules can then be used to determine the relation between F_Λ and the coordinates X^Λ . In components, the symplectic transformations are given by

$$\begin{aligned} X &\rightarrow \mathbf{U}X + \mathbf{V}\eta\partial F, \\ \partial F &\rightarrow -\eta\mathbf{V}X + \eta\mathbf{U}\eta\partial F, \end{aligned} \quad (13)$$

where X and ∂F are $(n+1)$ -dimensional complex vectors with components X^Λ and F_Λ respectively. It is clear from the transformation relations (13) that ∂F can be identified with $i\eta X$, in which case, a holomorphic function F exists and is given, in terms of the coordinates X , by

$$F = \frac{i}{2} X^t \eta X. \quad (14)$$

With the above relation, the complex vector X transforms as

$$X \rightarrow (\mathbf{U} + i\mathbf{V})X = \mathbf{M}X. \quad (15)$$

This transformation implies that the complex vector X should be identified with the complex vector parametrizing the $\frac{SU(1, n)}{SU(n) \otimes U(1)}$ coset. The set of complex

coordinates parametrizing this coset satisfy the following relation

$$\phi^\dagger \eta \phi = 1, \quad \text{where} \quad \phi = \begin{pmatrix} \phi^0 \\ \vdots \\ \phi^{n+1} \end{pmatrix}, \quad (16)$$

and are parametrized in terms of unconstrained coordinates z^α by [17]

$$\phi^0 = \frac{1}{\sqrt{Y}}, \quad \phi^j = \frac{z^\alpha}{\sqrt{Y}}, \quad \alpha = 1, \dots, n, \quad (17)$$

where $Y = 1 - \sum_\alpha z^\alpha \bar{z}^\alpha$. Here we identify X with the complex vector $\frac{1}{\sqrt{2}}\phi$. The special coordinates in this case are given by z^α and thus $Z^0 = 1$, $Z^\alpha = z^\alpha$, and the Kähler potential is given by

$$K = -\log(1 - \sum_\alpha z^\alpha \bar{z}^\alpha). \quad (18)$$

A different embedding of $SU(1, n)$ into $Sp(2n + 2, \mathbf{R})$ lead in general to a different relation between F_Λ and X^Λ . In fact once an embedding Ω_e is specified, then for symplectic transformations $S \in Sp(2n + 2, \mathbf{R})$, the matrix

$$\Omega'_e = S \Omega_e S^{-1}, \quad (19)$$

defines another embedding with a new section. As an example, consider the element $S_1 \in Sp(2n + 2, \mathbf{R})$, given by

$$S_1 = \begin{pmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \Sigma \end{pmatrix}, \quad (20)$$

where

$$\Sigma = \begin{pmatrix} \sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad \sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (21)$$

Using (19) and (20) we obtain another embedding of $SU(1, n)$ into $Sp(2n + 2, \mathbf{R})$

given by

$$\Omega'_e = \begin{pmatrix} \Sigma \mathbf{U} \Sigma & \Sigma \mathbf{V} \eta \Sigma \\ -\Sigma \eta \mathbf{V} \Sigma & \Sigma \eta \mathbf{U} \Sigma \end{pmatrix}. \quad (22)$$

For this embedding, we define a new symplectic section $(X', \partial F')$ which transforms as a vector under the action of Ω'_e . It is obvious that the two sections $(X, \partial F)$ and $(X', \partial F')$ are related by the following relations

$$X' = \Sigma X, \quad (\partial F)' = \Sigma \partial F. \quad (23)$$

This in components lead to

$$\begin{aligned} X'^0 &= \frac{1}{\sqrt{2}}(X^0 + X^1), \\ X'^1 &= \frac{1}{\sqrt{2}}(X^0 - X^1), \\ X'^j &= X^j, \quad j = 2, \dots, n, \\ F'_0 &= \frac{1}{\sqrt{2}}(F_0 + F_1) = \frac{i}{\sqrt{2}}(X^0 - X^1) = iX'^1, \\ F'_1 &= \frac{1}{\sqrt{2}}(F_0 - F_1) = \frac{i}{\sqrt{2}}(X^0 + X^1) = iX'^0, \\ F'_j &= F_j = -iX^j = -iX'^j. \end{aligned} \quad (24)$$

From these relations, it can be easily seen that there exists a holomorphic function F' which can be expressed in terms of X' by

$$F' = i \left(X'^0 X'^1 - \frac{1}{2} \sum_{j=2}^n (X'^j)^2 \right). \quad (25)$$

In this case

$$Z'^0 = 1, \quad Z'^1 = \frac{1 - z^1}{1 + z^1}, \quad Z'^j = \frac{\sqrt{2} z^j}{1 + z^1}, \quad (26)$$

and the Kähler potential is given by

$$K = -\log(Z'^1 + \bar{Z}'^1 - \sum_i Z'^i \bar{Z}'^i). \quad (27)$$

Formally, given two sections connected by a symplectic transformation

$$\begin{pmatrix} X' \\ \partial F' \end{pmatrix} = \Omega \begin{pmatrix} X \\ \partial F \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} X \\ \partial F \end{pmatrix}, \quad \Omega \in Sp(2n+2, \mathbf{R}), \quad (28)$$

then it can be shown that [1]

$$F' = \frac{1}{2} (X \quad \partial F) \begin{pmatrix} \mathbf{C}^t \mathbf{A} & \mathbf{C}^t \mathbf{B} \\ \mathbf{D}^t \mathbf{A} & \mathbf{D}^t \mathbf{B} \end{pmatrix} \begin{pmatrix} X \\ \partial F \end{pmatrix}. \quad (29)$$

Using (14),(29) and (20), the holomorphic function F' defined in (25) can be obtained.

Finally, we discuss the automorphic function of $CP(n-1, 1)$, which in physical terms might serve as a non-perturbative superpotential for the scalar fields, provided that the isometry group remains a nonperturbative symmetry. In [16] Ferrara *et al*, based on symmetry arguments, proposed an expression for the automorphic superpotential Δ for a special Kähler manifold with a symplectic section $(X, \partial F)$, defined by

$$\log ||\Delta||^2 = - \left[\sum_{M,N} \log |MX + N\partial F|^2 \right]_{\text{reg}}. \quad (30)$$

The constant vectors M, N define $Sp(2n+2, \mathbf{Z})$ orbits and transform in a conjugate way to that of $(X, \partial F)$ in order for the expression (30) to be invariant under the symplectic transformations.

Given the duality group Γ of a model with n physical scalar fields parametrizing a special Kähler manifold, the vector $(X, \partial F)$ transforms by

$$\begin{pmatrix} X \\ \partial F \end{pmatrix} \rightarrow \Omega_\Gamma \begin{pmatrix} X \\ \partial F \end{pmatrix} = \begin{pmatrix} \mathbf{A}' & \mathbf{B}' \\ \mathbf{C}' & \mathbf{D}' \end{pmatrix} \begin{pmatrix} X \\ \partial F \end{pmatrix}, \quad (31)$$

where the matrix Ω_Γ is the embedding of the duality Γ group in $Sp(2n+2, \mathbf{Z})$. For (30) to be invariant under the duality transformations, we get the following

transformation for the numbers (M, N)

$$\begin{pmatrix} M \\ N \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{D}' & -\mathbf{C}' \\ -\mathbf{B}' & \mathbf{A}' \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}. \quad (32)$$

For the case at hand, the symplectic transformation are those obtained by the embedding of $SU(1, n)$ into $Sp(2n+2, \mathbf{Z})$. Such an embedding can be given by Ω_e for which (32) gives the

following transformation

$$\begin{pmatrix} M \\ N \end{pmatrix} \rightarrow \begin{pmatrix} \eta \mathbf{U}' \eta & \eta \mathbf{V}' \\ -\mathbf{V}' \eta & \mathbf{U}' \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}, \quad (33)$$

where \mathbf{U}' and \mathbf{V}' are integer valued matrices.

Using the expression of the prepotential corresponding to this embedding as given in (14), eq. (30) gives the following expression for the automorphic superpotential

$$\log ||\Delta||^2 = - \left[\sum_{M'} \log |M'^t X|^2 \right]_{\text{reg}}, \quad (34)$$

where M' is an $(n+1)$ -dimensional complex vector with components given by

$$\begin{aligned} M'_0 &= M_0 + iN_0, \\ M'_i &= M_i - iN_i, \quad i = 1, \dots, n. \end{aligned} \quad (35)$$

Under the action of the symplectic transformations (33), we have

$$X \rightarrow \Omega X, \quad \Omega \in SU(1, n, \mathbf{Z}). \quad (36)$$

It can be easily shown that the following transformations for the quantum numbers hold,

$$\begin{pmatrix} M'_0 \\ -M'_i \end{pmatrix} \rightarrow \Omega^* \begin{pmatrix} M'_0 \\ -M'_i \end{pmatrix}. \quad (37)$$

Eq. (35) thus gives an explicit relation between the numbers M, N and those parametrizing $SU(1, n, \mathbf{Z})$ orbits. An explicit form of the automorphic function

can be obtained by using a regularization procedure such as the ζ -function regularization scheme [16,14].

To conclude, we have analysed the cosets $CP(n-1,1)$ with regard to the construction of the holomorphic function describing their special geometry. The knowledge of this function enables one to construct the lagrangian of the $N=2$ supergravity where the scalar fields of the vector supermultiplets parametrize the coset $CP(n-1,1)$. It should be mentioned, however, that our method always provides a relation between F_Λ and the coordinates X^Λ . It may occur that for a particular embedding an F function does not exist. As a matter of fact, in some physically interesting cases, one needs a formulation of the theory in which an F function does not exist. This is the case in the study of perturbative corrections of vector coupling in $N=2$ heterotic string vacua [18,19], and in the study of supersymmetry breaking of $N=2$ supersymmetry down to $N=1$ [20]. In this case, the physical quantities in the new basis, are obtained by performing a symplectic transformation on the physical quantities in a system of coordinates where an F function exists.

ACKNOWLEDGEMENT

The work of W. A. Sabra is supported by PPARC, S. Thomas is supported by the Royal Society of Great Britain, and N. Vanegas is funded by Colciencias and U. de Antioquia.

REFERENCES

1. B. de Wit and A. Van Proeyen, *Special geometry and symplectic transformation*, lectures delivered by A. Van Proeyen at the String workshop on String theory. Trieste, April 1994, hep-th/9510186.
2. B. de Wit, P. G. Lauwers, R. Philippe, Su S. Q and A. Van Proeyen, *Phys. Lett.* **B134** (1984) 37; B. de Wit and A. Van Proeyen, *Nucl. Phys.* **B245** (1984) 89.
3. E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, *Nucl. Phys.* **B250** (1985) 385.
4. B. de Wit, P. G. Lauwers and A. Van Proeyen, *Nucl. Phys.* **B255** (1985) 569.
5. E. Cremmer and A. Van Proeyen, *Class. Quantum Grav.* **2** (1985) 445.
6. S. Ferrara and A. Van Proeyen, *Class. Quantum Grav.* **6** (1989) 124.
7. A. Strominger, *Commun. Math. Phys.* **133** (1990) 163.
8. L. Castellani, R. D'Auria and S. Ferrara, *Phys. Lett.* **B241** (1990) 57.
9. R. D'Auria, S. Ferrara and P. Frè, *Nucl. Phys.* **B359** (1991) 705.
10. S. Cecotti, S. Ferrara, and L. Girardello, *Int. J. Mod. Phys.* **4** (1989) 2475; *Phys. Lett.* **B213** (1988) 443.
11. S. Ferrara and A. Strominger, in *Strings '89*, ed. R. Arnowitt, R. Bryan, M. J. Duff, D. Nanopoulos and C. N. Pope, World Scientific, Singapore, (1990) 245.
12. L. Dixon, V. Kaplunovsky and J. Louis, *Nucl. Phys.* **B329** (1990) 27.
13. P. Candelas and X. de la Ossa, *Nucl. Phys.* **B355** (1991) 455; P. Candelas, X. de la Ossa, P. Green and L. Parkers, *Nucl. Phys.* **B359** (1991) 21.
14. P. Frè, P. Soriani, *Nucl Phys* **B371** (1992) 659.
15. M. K. Gaillard and B. Zumino, *Nucl. Phys.* **B193** (1981) 221.

- 16. S. Ferrara, C. Kounnas, D. Lüst and F. Zwirner, *Nucl. Phys.* **B365** (1991) 431.
- 17. G. L. Cardoso, D. Lüst and T. Mohaupt, *Nucl. Phys.* **B450** (1995) 115.
- 18. A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *Nucl. Phys.* **B444** (1995) 92.
- 19. B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, *Nucl. Phys.* **B451** (1995) 53; I. Antoniadis, S. Ferrara, G. Gava, K. S. Narain and T. R. Taylor, *Nucl. Phys.* **B447** (1995) 35.
- 20. S. Ferrara, L. Girardello and M. Porrati, hep-th/9510074.